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KAEHLER STRUCTURES ON TORIC VARIETIES

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1. Let (X, ω) be a compact connected $2n$-dimensional manifold, and let

$$\tau\colon T^n \to \mathrm{Diff}(X, \omega) \tag{1.1}$$

be an effective Hamiltonian action of the standard n-torus. Let $\phi\colon X \to \mathbb{R}^n$ be its moment map. The image, Δ, of ϕ is a convex polytope, called the *moment polytope.* Delzant showed in [5] that the triple (X, ω, τ) is determined up to isomorphism by this polytope, and also that X has an intrinsic T^n-invariant complex structure which is compatible with ω and makes X into a toric variety. The purpose of this note is to show that is not only the symplectic geometry of X determined by Δ, but also, to a certain extent, the *Kaehler* geometry of X. By [5], Δ can be described by a set of inequalities of the form

$$\langle x, u_i\rangle \geq \lambda_i, \qquad i = 1, \cdots, d; \tag{1.2}$$

the u_i's being primitive elements of the lattice, $\mathbb{Z}^n$, and d the number of $(n-1)$-dimensional faces of Δ. Let $l_i\colon \mathbb{R}^n \to \mathbb{R}$ be the map

$$l_i(x) = \langle x, u_i\rangle - \lambda_i,$$

and let Δ° be the interior of Δ. Then $x \in \Delta^\circ$ if and only if $l_i(x) > 0$ for all i. Let

$$l_\infty(x) = \sum_{i=1}^{d} \langle x, u_i\rangle.$$

Our main result is the following formula for the restriction of ω to $\phi^{-1}(\Delta^\circ)$:

$$\omega = \sqrt{-1}\partial\bar{\partial}\pi^*\left(\sum_{i=1}^{d} \lambda_i(\mathrm{Log}\, l_i) + l_\infty\right). \tag{1.3}$$

This we will derive as a corollary of another result which I will now describe: By [5] there is an intrinsic involution $\gamma\colon X \to X$ which reverses

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the complex structure and maps ω to $-\omega$. Let X_r be the fixed point set of γ. The restriction of ϕ to X_r is a ramifed cover

$$\psi: X_r \to \Delta \tag{1.4}$$

which breaks into 2^n connected components over Δ°. These components are mapped diffeomorphically onto Δ° by ψ, and each of them has a Riemannian metric (as a submanifold of the Kaehler manifold, X). We will show that (1.3) is a consequence of the following:

Theorem. *Let X_r^ε, $\varepsilon = (\pm 1, \cdots, \pm 1)$, be one of the 2^n connected components of $\psi^{-1}(\Delta^\circ)$. Then the metric on X_r^ε described above is the pull-back to ψ of the metric*

$$\frac{1}{2}\sum_{i=1}^{d}\frac{(dl_i)^2}{l_i} \tag{1.5}$$

on Δ°.

The submanifold of X defined by $l_i \circ \phi = 0$ is a complex submanifold of codimension 2. Let c_i be the cohomology class in $H^2(X, \mathbb{Z})$ dual to this manifold. As a corollary of (1.3) we will show that

$$\frac{1}{2\pi}[\omega] = -\sum_{i=1}^{d}\lambda_i c_i. \tag{1.6}$$

Now let ω_{FS} be the standard $SU(N+1)$-invariant Kaehler form on $\mathbb{C}P^N$, normalized so that $[\omega_{FS}]/(2\pi)$ is the oriented generator of $H^2(\mathbb{C}P^N, \mathbb{Z})$. It is known that there exist, for some N, a linear representation, $\rho: T^n \to SU(N+1)$, and a projective embedding $\imath: X \to \mathbb{C}P^N$ which intertwines the action of T^n on X with the action of T^n induced by ρ. Moreover, if the λ_i's are integers,[1] one can require this embedding to satisfy

$$\imath^\#[\omega_{FS}] = [\omega], \tag{1.7}$$

which implies, by a thoerem of Banyaga, that ω and $\imath^*\omega_{FS}$ are T^n-equivariantly symplectomorphic. In §5 we will prove the following refined version of (1.7):

$$\imath^*\omega_{FS} = \omega + i\partial\bar{\partial}\phi^*(-l_\infty + \operatorname{Log} P), \tag{1.8}$$

P being the polynomial

$$\sum_\alpha k_\alpha \prod_{i=1}^{d} l_i^{n_{i\alpha}}, \qquad \alpha \in \mathbb{Z}^n \cap \Delta. \tag{1.9}$$

[1]Or alternatively, if the vertices of Δ are in $\mathbb{Z}^n$.

Here $n_{i\alpha} = l_i(\alpha)$, and k_α is a nonnegative constant which depends on the embedding, $\imath$, and can be fairly arbitrary. However, we will show that for all embeddings, k_α has to be positive if α is a vertex of Δ. (This turns out to be a necessary and sufficient condition for P to be positive on Δ and hence for (1.7) to be well-defined.)

The following is a summary of the contents of this article: §2 contains a brief sketch of the theory of toric varieties in the spirit of [2, Chapter VI]. The above theorem will be proved in §3, which will be used to derive formula (1.3). In §5 we will make some general comments about projective embeddings of X and prove (1.7)–(1.8). Finally in §6 we will derive (1.6), and also give two applications of (1.6). The first application is a new proof of the "combinatorial Riemann-Roch" formula of Khovanskii and Pukhlikov [11], and the second a generalization of a well-known result on the topology of toric surfaces: As above let X_i be the codimension-2 submanifold of X defined by $l_i \circ \phi = 0$. If $\dim X = 4$, the intersection numbers $\#(X_i \cap X_j)$ are given by a simple formula involving the angles of the moment polygon. (See, for instance [2, p. 176].) We will describe in §6 what the analogous formula is in n dimensions.

We would like to thank Shlomo Sternberg for convincing us of the importance of getting a closed form expression for the above Kaehler metric, and also to thank Jean-Michel Kantor for several helpful discussions about the Khovanskii-Pukhlikov results (and some very exciting generalizations of those results by Khovanskii and himself which are described in [10]).

2. Let Δ be a convex polytope in $\mathbb{R}^n$ defined by a system of inequalities of the form

$$(2.1) \qquad \langle x, u_i \rangle \geq \lambda_i, \qquad i = 1, \cdots, d,$$

where u_i is the inward-pointing normal vector to the ith $(n-1)$-dimensional face of Δ, and d is the number of the $(n-1)$-dimensional faces. We will assume that these normal vectors are *rational*, in which case we can normalize u_i by requiring it to be a primitive element of the integer lattice, $\mathbb{Z}^n$. We will say that Δ is *n-valent* if these are exactly n edges intersecting in each vertex, p, and that Δ is nonsingular at p if there exists a basis, $\{w_1, \cdots, w_n\}$, of $\mathbb{Z}^n$ such that the n edges meeting at p lie on the rays, $p + tw_i$, $0 \leq t < \infty$. If Δ is nonsingular at all vertices, we will simply say that Δ is *nonsingular.* Let Δ be a polytope with this property. Our goal in this section is to show that there exist a compact $2n$-dimensional Kaehler manifold (X, ω) and a Hamiltonian action $\tau\colon T^n \to \mathrm{Diff}(X, \omega)$ for which Δ is the moment polytope. Here are the details.

I. Let $(e_1, \cdots, e_d)$ be the standard basis vectors of $\mathbb{R}^d$ and let

$$\beta: \mathbb{R}^d \to \mathbb{R}^n \tag{2.2}$$

be the mapping which maps e_i onto u_i. Let n be the kernel of β. It is clear from the hypotheses on Δ that β is surjective; so one gets an exact sequence:

$$0 \to n \xrightarrow{\imath} \mathbb{R}^d \xrightarrow{\beta} \mathbb{R}^n \to 0, \tag{2.3}$$

and by duality an exact sequence:

$$0 \to \mathbb{R}^n \xrightarrow{\beta^*} \mathbb{R}^d \xrightarrow{\imath^*} n^* \to 0. \tag{2.4}$$

Let

$$T^n_{\mathbb{C}} = \mathbb{C}^n / 2\pi i \mathbb{Z}^n \quad \text{and} \quad T^d_{\mathbb{C}} = \mathbb{C}^d / 2\pi i \mathbb{Z}^d.$$

The map (2.2) extends to a map $\beta_{\mathbb{C}}: \mathbb{C}^d \to \mathbb{C}^n$ which maps $2\pi i \mathbb{Z}^d$ onto $2\pi i \mathbb{Z}^n$ and hence induces a mapping

$$\beta_{\mathbb{C}}: T^d_{\mathbb{C}} \to T^n_{\mathbb{C}}. \tag{2.5}$$

Let $N_{\mathbb{C}}$ be the kernel of this mapping. Then, corresponding to (2.3), one has an exact sequence of complex groups:

$$0 \to N_{\mathbb{C}} \to T^d_{\mathbb{C}} \to T^n_{\mathbb{C}} \to 0. \tag{2.6}$$

II. Now let $\kappa: T^d_{\mathbb{C}} \to GL(d, \mathbb{C})$ be the linear action of $T^d_{\mathbb{C}}$ on $\mathbb{C}^d$ defined by

$$\kappa(w)z = ((\exp w_1)z_1, \cdots, (\exp w_d)z_d), \tag{2.7}$$

and let κ_1 be the restriction of κ to N. The space, X, which we are after, is roughly speaking the quotient of $\mathbb{C}^d$ by the action, κ_1, or, in other words, the set of $N_{\mathbb{C}}$-orbits in $\mathbb{C}^d$. However, for this orbit space to be nonsingular, we will have to delete from it the "unstable" $N_{\mathbb{C}}$-orbits. This is done as follows: For every multi-index

$$I = (i_1, \cdots, i_r), \qquad 1 \le i_1 < \cdots < i_r \le d,$$

let

$$\mathbb{C}^d_I = \{(z_1, \cdots, z_d),\ z_i = 0 \text{ iff } i \in I\}. \tag{2.8}$$

It is clear that (2.8) is a $T^d_{\mathbb{C}}$ orbit, and also that *every* $T^d_{\mathbb{C}}$ orbit is of this type; so (2.8) sets up a one-one correspondence between $T^d_{\mathbb{C}}$ orbits in $\mathbb{C}^d$ and multi-indices. Now let F be a face of Δ of codimension r. Then, by (2.1), F is defined by a system of equalities, $\langle x, u_i \rangle = \lambda_i$, $i \in I$, where

I is a multi-index of length r. We will define: $\mathbb{C}_F^d = \mathbb{C}_I^d$. (For instance, if F is the open face of Δ, then $\mathbb{C}_F^d = \{(z_1, \cdots, z_d); z_i \neq 0 \text{ for all } i\}$. In other words, $\mathbb{C}_F^d$ is the open $T_{\mathbb{C}}^d$ orbit in $\mathbb{C}^d$.) Now let

$$\mathbb{C}_\Delta^d = \bigcup_F \mathbb{C}_F^d. \tag{2.9}$$

If $\mathbb{C}_I^d$ is *not* in the union, (2.9), then, by (2.8), the $T_{\mathbb{C}}^d$-orbits in the closure of $\mathbb{C}_I^d$ also are not in this union; so (2.9) is an open subset of $\mathbb{C}^d$. Moreover, since it is a union of $T_{\mathbb{C}}^d$ orbits, it is stable under the action of $N_{\mathbb{C}}$. For the following see [2].

Theorem 2.1. *$N_{\mathbb{C}}$ acts freely and properly on $\mathbb{C}_\Delta^d$, and the orbit space*

$$\mathbb{C}_\Delta^d / N_{\mathbb{C}} \tag{2.10}$$

is a compact manifold.

We will denote this orbit space by X. It has, by definition, a natural complex structure. Moreover, by the exact sequence (2.6),

$$T_{\mathbb{C}}^n \simeq T_{\mathbb{C}}^d / N_{\mathbb{C}};$$

so there is a natural action of $T_{\mathbb{C}}^n$ on X, and the orbits of this action are the images of the orbits of $T_{\mathbb{C}}^d$ in $\mathbb{C}_\Delta^d$. However, by (2.9) these orbits are the sets, $\mathbb{C}_F^d$; so their images are

$$\mathbb{C}_F^d / N_{\mathbb{C}}. \tag{2.11}$$

Thus the $T_{\mathbb{C}}^n$ orbits in X are in one-one correspondence with the faces of Δ. The orbit corresponding to the open face is the only open orbit and on this orbit $T_{\mathbb{C}}^n$ acts freely.

III. Let $\sigma: \mathbb{C}^d \to \mathbb{C}^d$ be the involution sending z to $\overline{z}$. Then by (2.7),

$$\kappa(a)(\sigma(z)) = \sigma(\kappa(\overline{a})(z)),$$

so σ maps the $N_{\mathbb{C}}$ orbit through z into the $N_{\mathbb{C}}$ orbit through $\sigma(z)$, and hence induces an involution

$$\gamma: X \to X, \tag{2.12}$$

on the orbit space X. We will denote by X_r the fixed point set of γ and refer to X_r from now on as the *real part* of X.

IV. We will now give an alternative description of X due to Atiyah [1] and Delzant [5]: Let T^d, T^n and N be the maximal compact subgroups of $T_{\mathbb{C}}^d$, $T_{\mathbb{C}}^n$, and $N_{\mathbb{C}}$ respectively. From (2.6) one gets an exact sequence

$$0 \to N \to T^d \to T^n \to 0, \tag{2.13}$$

and from (2.7) an action of T^d on $\mathbb{C}^d$ which preserves the symplectic form:

(2.14) $$\frac{i}{2}\sum_{k=1}^{d} dz_k \wedge d\overline{z}_k.$$

This action is Hamiltonian, and its moment map is the map

(2.15) $$h(z) = \frac{1}{2}\sum_{k=1}^{d} |z_k|^2 e_k + c,$$

c being an arbitrary constant. Following [5] we will set c equal to

(2.16) $$\sum_{k=1}^{d} \lambda_k e_k$$

the λ_k's being the λ_k's in (2.1). Restricting to N one gets a Hamiltonian action of N on $\mathbb{C}^d$ with moment map

(2.17) $$f(z) = \frac{1}{2}\sum_{k=1}^{d} |z_k|^2 \alpha_k + \lambda,$$

where $\alpha_k = \iota^* e_k$ and $\lambda = \sum \lambda_k \alpha_k$. (See (2.4).) We will denote the zero level set of this moment map by Z, noting that, by (2.17), Z is defined by the quadratic equation:

(2.18) $$\frac{1}{2}\sum |z_k|^2 \alpha_k = -\lambda.$$

In [5] Delzant proves:

Theorem 2.2. *Z is a compact submanifold of $\mathbb{C}^d$, and the action of N on Z is free. Hence the quotient space*

(2.19) $$X = Z/N$$

is a compact manifold.

Let

(2.20) $$\pi : Z \to X$$

be the projection of Z onto X and let

(2.21) $$\iota : Z \to \mathbb{C}^d$$

be the inclusion map. Then, by [13], there is a canonical symplectic form, ω, on X with the property:

(2.22) $$\pi^* \omega = \iota^*((i/2)\sum dz_k \wedge d\overline{z}_k).$$

Moreover, the action of T^d on $\mathbb{C}^d$ leaves invariant the map, (2.15), and hence Z by (2.18). Thus, by (2.19) there is an induced action of the quotient group, $T^n = T^d/N$, on X. It is easy to see, by staring at (2.22), that this has to be a Hamiltonian action. To see what its moment map is, we note that by (2.4) and (2.18):

$$h \circ \iota = \beta^* \circ g, \tag{2.23}$$

ι being the inclusion mapping, (2.21), β^* the transpose of (2.2), and g a mapping of Z into $\mathbb{R}^n$. Since h is T^d-invariant, g has to be N-invariant by (2.23); so there exists a mapping

$$\phi: X \to \mathbb{R}^n \tag{2.24}$$

satisfying

$$\phi \circ \pi = g, \tag{2.25}$$

and we claim that *this* is the moment map associated with the Hamiltonian action of T^n on X. (We will not, however, bother to prove this.) For the following see [5].

Theorem 2.3. *The image of* ϕ *is* Δ. *Hence* Δ *is the moment polytope associated with the action of* T^n *on* X.

V. We now have two definitons of X, namely (2.10) and (2.19). That these two definitions are consistent follows from the following result of Audin:

Theorem 2.4. Z *is contained in* $\mathbb{C}^d_\Delta$, *and every* $N_{\mathbb{C}}$*-orbit in* C^d_Δ *intersects* Z *in an* N*-orbit.*

See [2, Chapter 6, Proposition 3.1]. Audin proves Theorem 2.4 by deducing it from a fairly deep result of Kirwan. For an elementary proof see the appendix to [7].

Finally we claim:

Theorem 2.5. *The symplectic structure on* X *is compatible with its complex structure. In other words, the form defined by* (2.22) *is Kaehler.*

Proof. See ([8], Theorem 3.5).

3. Theorem 2.5 implies that X has an intrinsic Kaehler metric, and our goal is to show that the restriction of this Kaehler metric to the open $T^n_{\mathbb{C}}$ orbit in X is given by (1.3). As an intermediate step in the proof of (1.3) we will show that this Kaehler metric induces a Riemannian metric on the real part, X_r, of X which can be explicitly computed in terms of "moment" data. Let $\sigma: \mathbb{C}^d \to \mathbb{C}^d$ be the involution, $\sigma(z) = \bar{z}$. By (2.18) the set Z is stable under σ. Let Z_r be the fixed point set of the restriction of σ to Z. From (2.20) one gets a map:

$$\pi\colon Z_r \to X_r, \tag{3.1}$$

with the following properties (whose verification we will leave as an exercise):

Theorem 3.1. *The map* (3.1) *is a* 2^{d-n}*-fold covering map, its group of deck transformations being the group*

$$\{a \in N,\ a^2 = 1\}. \tag{3.2}$$

A point, $z = x + iy$, of $\mathbb{C}^d$ is a fixed point of σ if and only if $y = 0$. Therefore, by (2.18), Z_r is the subset of $\mathbb{R}^d$ defined by the quadratic equation

$$\sum_{i=1}^{d} x_i^2 \alpha_i = -\lambda. \tag{3.3}$$

We will equip Z_r with a Riemannian metric by restricting to (3.3) the standard Euclidean metric on $\mathbb{R}^d$ (or, alternatively, the standard Kaehler metric on $\mathbb{C}^d$). Similarly we wil equip X_r with a Riemannian metric by restricting to it the Kaehler metric on X.

Theorem 3.2. *The covering map* (3.1) *is an isometry with respect to these two metrics.*

Proof. Let p be an arbitrary point of Z_r and let q be its image in X_r. Let T_p^{vert} be the tangent space to the $N_{\mathbb{C}}$ orbit in $\mathbb{C}^d$ containing p and let T_p^{hor} be its orthocomplement with respect to the Kaehler metric on $T_p\mathbb{C}^d$. The following are easy to check:

1. T_p^{hor} is tangent to Z at p.

2. $d\sigma_p$ maps the spaces, T_p^{vert} and T_p^{hor}, into themselves. (Since $d\sigma_d$ preserves the Kaehler metric on $T_p\mathbb{C}^d$, it suffices to check this for T_p^{vert}. But since $\sigma(p) = p$, σ maps the $N_{\mathbb{C}}$ orbit through p into itself.)

3. The map $d\pi_p\colon T_pZ \to T_qX$ maps T_p^{hor} bijectively onto T_qX.

4. The tangent space to Z_r at p is the fixed point set of the map: $d\sigma_p\colon T_p^{\text{hor}} \to T_p^{\text{hor}}$.

5. $d\pi_p$ maps T_pZ_r bijectively onto T_qX.

Identify $T_p\mathbb{C}^d$ with $\mathbb{C}^d$ and let $J_p\colon T_p\mathbb{C}^d \to T_p\mathbb{C}^d$ be the mapping, "multiplication by $\sqrt{-1}$."

6. J_p maps T_p^{hor} and T_p^{vert} into themselves. (Since J_p preserves the Kaehler metric on $T_p\mathbb{C}^d$ it suffices to check this for T_p^{vert}. But T_p^{vert} is

the tangent space to a complex submanifold of $\mathbb{C}^d$.)

7. The bijective map:

$$d\pi_p : T_p^{\text{hor}} \to T_q X \tag{3.4}$$

intertwines J_p with a mapping, J_q, of $T_q X$ into itself; and this mapping is the defining mapping for the complex structure on X at q.

8. T_p^{hor} is a symplectic subspace of $T_p\mathbb{C}^d$, and (3.4) is a symplectic mapping.

From items 6–8 one concludes:

9. The map (3.4) maps the Kaehler metric on T_p^{hor} onto the Kaehler metric on $T_q X$.

Finally by item 9 and items 2–5, it is clear that (3.1) is an isometry at p. q.e.d.

Now let us restrict the Euclidean metric

$$\sum_{i=1}^{d} (dx_i)^2 \tag{3.5}$$

to one of the 2^d open orthants:

$$\varepsilon_1 x_1 > 0, \cdots, \varepsilon_d x_d > 0, \qquad \varepsilon_i = \pm 1. \tag{3.6}$$

If one makes the coordinate change

$$s_i = x_i^2/2, \qquad i = 1, \cdots, d, \tag{3.7}$$

then, on the set (3.6), the metric (3.5) becomes

$$\frac{1}{2}\sum_{i=1}^{d} \frac{(ds_i)^2}{s_i}. \tag{3.8}$$

Let Z_r^ε be the intersection of Z_r with the set (3.6). In the s-coordinates Z_r^ε is just the intersection of the positive orthant $s_1 > 0, \cdots, s_d > 0$ with the linear space

$$\sum s_i \alpha_i = -\lambda \tag{3.9}$$

in view of (3.3). Moreover, the moment map (2.15) becomes a linear mapping

$$s \to \sum (s_i + \lambda_i) e_i \tag{3.10}$$

in the s-coordinates. Let $h = h(s)$ be the restriction of this mapping to Z_r^ε. We will prove that h maps Z_r^ε diffeomorphically onto a subset of $\mathbb{R}^d$ which can be naturally identified with the interior of Δ: By (2.4) one has an exact sequence $0 \to \mathbb{R}^n \xrightarrow{\beta^*} \mathbb{R}^d \xrightarrow{\iota^*} n^* \to 0$. Moreover, $\alpha_i = \iota^* e_i$ by definition; so by (3.9) and (3.10), $\iota^* h(s) = 0$ for all $s \in Z_r^\varepsilon$. Hence, by this exact sequence: $h(s) = \beta^* x$ for some $x \in \mathbb{R}^n$. However, by (3.10),

$$\langle \beta^* x, e_i \rangle = s_i + \lambda_i > \lambda_i, \tag{3.11}$$

since $s_i > 0$. On the other hand,

$$\langle \beta^* x, e_i \rangle = \langle x, \beta e_i \rangle = \langle x, u_i \rangle, \tag{3.12}$$

so $\langle x, u_i \rangle > \lambda_i$; and hence, by (2.1), x is in the interior, Δ^0, of Δ. Thus we have proved the following:

Theorem 3.3. *The mapping, h, is a diffeomorphism of Z_r^ε onto $\beta^* \Delta^\circ$.*

Since β^* is injective, $\Delta^\circ \simeq \beta^* \Delta^\circ$; so we can regard h as a diffeomorphism of Z_r^ε onto Δ°.

Let $l_i \colon \Delta^\circ \to \mathbb{R}$ be the linear function

$$l_i(x) = \langle x, u_i \rangle - \lambda_i.$$

We will make Δ° into a Riemannian manifold by equipping it with the metric

$$\frac{1}{2} \sum_{i=1}^d \frac{(dl_i)^2}{l_i}. \tag{3.13}$$

By (3.11) and (3.12),

$$l_i \circ h = s_i; \tag{3.14}$$

so the pull-back to X_r^ε of the metric (3.13) by h is equal to the restriction to X_r^ε of the Euclidean metric, (3.8).

Let X_r^ε be the image of Z_r^ε in X with respect to the mapping (3.1), and let ψ be the restriction to X_r^ε of the moment mapping (2.24). Then by (2.25) $\psi \circ \pi = h$; and, therefore, by Theorem 3.1, one gets the following description of the Riemannian metric induced on X_r^ε by the Kaehler metric on X:

Theorem 3.4. *Under the moment map $\psi \colon X_r^\varepsilon \xrightarrow{\simeq} \Delta^\circ$, the Riemannian metric on X_r^ε is mapped onto the metric* (3.13).

4. For the proof of (1.3) we will need a few elementary facts about Kaehler structures on complex tori. Let M be the complex torus $\mathbb{C}^n / 2\pi i \mathbb{Z}^n$ and let T^n act on M by the action:

$$T^n \times M \to M, \qquad (y, z) \to z + \sqrt{-1} y. \tag{4.1}$$

Let ω be a Kaehler form on M which is T^n-invariant.

Theorem 4.1. *ω is exact $\Leftrightarrow$ the action* (4.1) *is Hamiltonian.*

Proof. The implication "$\Rightarrow$" is a standard result. (See [8, §26].) To prove the implication the other way it suffices to note that if W is a T^n orbit, the inclusion map of W into M is an isomorphism on cohomology. However, if the action (4.1) is Hamiltonian, W is a Lagrangian submanifold of M [8, 188], so the pull-back of ω to W vanishes.

Theorem 4.2. *The invariant Dolbeault cohomology groups $H^{o,i}(M)_{T^n}$ are zero for $i > 0$.*

Proof. An invariant Dolbeault k-form is of the form

$$\omega = \sum f_I(x) d\overline{z}_I, \qquad f_I \in C^\infty(\mathbb{R}^n),$$

summed over multi-indices, I, of length k. Let $\imath: \mathbb{R}^n \to M$ be the inclusion map. Then

$$\imath^*\omega = \sum f_I(x)\, dx_I;$$

so the map $\imath^*: \Omega^{o,k}(M)_{T^n} \to \Omega^k(\mathbb{R}^n)$ is bijective. It is clear, moreover, that $\imath^*\overline{\partial} = d\imath^*$; so $\imath^*$ induces a bijective map on cohomology: $\imath^*: H^{o,k}(M)_{T^n} \to H^k(\mathbb{R}^n)$. q.e.d.

We will need the following two results:

Theorem 4.3. *Let ω be a T^n-invariant Kaehler form on M. Then the action of T^n on (M, ω) is Hamiltonian if and only if ω possesses a T^n-invariant potential function, i.e., if and only if there exists a function, $F \in C^\infty(\mathbb{R}^n)$, with the property*

$$\omega = 2i\partial\overline{\partial}F. \tag{4.2}$$

Proof. If the action (4.1) is Hamiltonian, there exists a T^n-invariant one form, ν, for which $\omega = d\nu$. Let $\nu = \alpha + \overline{\alpha}$ where $\alpha \in \Omega^{0,1}$. Then $\overline{\partial}\alpha = \partial\alpha = 0$ and

$$\omega = d\nu = \partial\alpha + \overline{\partial}\overline{\alpha}. \tag{4.3}$$

By Lemma 2 there exists a T^n-invariant function, G, for which $\alpha = \overline{\partial}G$ and hence, by (4.3),

$$\omega = \partial\overline{\partial}G + \overline{\partial}\partial\overline{G} = 2i\partial\overline{\partial}(\operatorname{Im} G). \quad \text{q.e.d.}$$

Assume the action of T^n on M is Hamiltonian, and let ϕ: ; $M \to \mathbb{R}^n$ be the moment map associated with this action. The other result we will need is:

Theorem 4.4. *Up to an additive constant, ϕ is the Legendre transform associated with F; i.e.,*

$$\phi(x + iy) = \partial F/\partial x + c, \qquad c \in \mathbb{R}^n. \tag{4.4}$$

Proof. By definition,

$$d\phi^k = -\iota(\partial/\partial y_k)\omega.$$

However, by (4.2),

$$\omega = \sum_{j,k=1}^{n} \frac{\partial^2 F}{\partial x_j \partial x_k} dx_j \wedge dy_k;$$

so

$$d\phi_k = -\iota\left(\frac{\partial}{\partial y_k}\right)\omega = d\left(\frac{\partial F}{\partial x_k}\right).$$

Thus $\phi_k = \partial F/\partial x_k + c_k$, for some constant c_k.

Remark. One can eliminate c by replacing F by $F - \sum_{k=1}^{n} c_k x_k$ (which does not change the Kaehler form (4.2)).

In the application we will make use of these results. We will take M to be the open $T_{\mathbb{C}}^n$ orbit in X, ω the restriction of the Kaehler form on X, and $\phi: M \to \mathbb{R}$ the restriction of the moment map associated with the action (1.1). By Theorem 4.3, ω can be written in the form

$$\frac{i}{2}\sum_{j,k=1}^{n} \frac{\partial^2 F}{\partial x_j \partial x_k} dz_j \wedge d\bar{z}_k, \tag{4.5}$$

and by Theorem 4.4 we can normalize F so that it has the property:

$$\phi(x+iy) = \frac{\partial F}{\partial x}(x). \tag{4.6}$$

The real part, X_r, of X intersects M in the set $\mathbb{R}^n \oplus \pi i(\mathbb{Z}^n/2\mathbb{Z}^n)$; in particular, $\mathbb{R}^n$ is a connected component of this set. The restriction to $\mathbb{R}^n$ of the Kaehler metric, (4.5), is the Riemannian metric

$$\sum \frac{\partial^2 F}{\partial x_j \partial x_k} dx_j\, dx_k, \tag{4.7}$$

and by Theorem 3.3, this is the pull-back by ϕ of the metric (1.5). This fact will enable us to obtain an explicit formula for F in terms of moment data, and hence by (4.5) an explicit formula for ω. For this purpose we first note that the metric (1.5) can be rewritten in the form

$$\sum \frac{\partial^2 G}{\partial y_j \partial y_k} dy_k\, dy_k, \tag{4.8}$$

where

$$G = \frac{1}{2}\sum_{k=1}^{d} l_k(y) \operatorname{Log} l_k(y). \tag{4.9}$$

Now let $\sum_{j=1}^{n} dx_i \wedge dy_i$ be the standard symplectic form on $\mathbb{R}^{2n}$, and let

$$\Gamma = \{(x, y) \in \mathbb{R}^{2n}, y = \partial F/\partial x\}$$

be the graph of the Legendre transform

$$x \in \mathbb{R}^n \to \partial F/\partial x \in \Delta^\circ. \tag{4.10}$$

Γ is a Lagrangian submanifold of $\mathbb{R}^{2n}$; since (4.10) is a diffeomorphism, the differentials $dx_1, \cdots, dx_n$ and $dy_1, \cdots, dy_n$ are independent on Γ. Moreover, by Theorem 3.3, the restriction to Γ of the quadratic differential

$$\sum dx_i\, dy_i \tag{4.11}$$

can be written as either

$$\sum \frac{\partial^2 F}{\partial x_i \partial x_j} dx_i\, dx_j \tag{4.12}$$

or

$$\sum \frac{\partial^2 G}{\partial y_i \partial y_j} dy_i\, dy_j. \tag{4.13}$$

However, setting $x = x(y)$ we can also write the restriction of (4.11) to Γ in the form

$$\frac{1}{2} \sum \left(\frac{\partial x_i}{\partial y_j} + \frac{\partial x_j}{\partial y_i} \right) dy_1\, dy_j. \tag{4.14}$$

Hence, comparing (4.13) with (4.14) yields

$$\frac{1}{2} \left(\frac{\partial x_i}{\partial y_j} + \frac{\partial x_j}{\partial y_i} \right) = \frac{\partial^2 G}{\partial y_i \partial y_j}.$$

However, since Γ is Lagrangian,

$$\frac{\partial x_i}{\partial y_j} - \frac{\partial x_j}{\partial y_i} = 0,$$

so

$$\frac{\partial x_i}{\partial y_j} = \frac{\partial^2 G}{\partial y_j \partial y_i},$$

i.e.

$$x_i = \partial G/\partial y_i + a_i, \qquad i = 1, \cdots, n, \tag{4.15}$$

a_i being a constant. Setting $H = G + \sum a_i y_i$ we conclude:

$$x = \partial H/\partial y. \tag{4.16}$$

In other words *the Legendre transform* (4.10) *is the inverse of the Legendre transform* (4.16). Since these two Legendre transforms are inverses of each other, F has to be the Legendre function dual to H, i.e.,

$$F(x) = \sum_{i=1}^{n} x_i y_i - H(y), \tag{4.17}$$

evaluated at $y = \partial F/\partial x$. As a function of y, however, the right-hand side of (4.17) is equal to:

$$\sum y_i \frac{\partial H}{\partial y_i} - H(y) \tag{4.18}$$

or

$$\sum y_i \frac{\partial G}{\partial y_i} - G(y), \tag{4.19}$$

since G differs from H by a function which is linear in y. Since $G = \frac{1}{2}\sum l_k(y)\mathrm{Log}\, l_k(y)$, the first term in (4.19) is equal to

$$\frac{1}{2}\sum_{i=1}^{n}\left(\sum_{k=1}^{d} y_i\left(\frac{\partial l_k}{\partial y_i}\mathrm{Log}\, l_k + \frac{\partial l_k}{\partial y_i}\right)\right);$$

and since $l_k(y) = \langle y, u_k\rangle - \lambda_k$, this can be written as

$$\frac{1}{2}\left(\sum (l_k + \lambda_k)\mathrm{Log}\, l_k + l_\infty\right) \tag{4.20}$$

where

$$l_\infty(y) = \langle y, u\rangle; \qquad u = \sum_{k=1}^{d} u_k. \tag{4.21}$$

Subtracting (4.9) from (4.20) we get for F:

$$\frac{1}{2}\phi^*\left(\sum_{k=1}^{d} \lambda_k \mathrm{Log}\, l_k + l_\infty\right); \tag{4.22}$$

so to summarize we have proved the following:

Theorem 4.5. *On the open $T_{\mathbb{C}}^n$ orbit in X, the Kaehler form, ω, is equal to*

$$i\partial\bar{\partial}\phi^*\left(\sum_{k=1}^{d} \lambda_k(\mathrm{Log}\, l_k) + l_\infty\right).$$

Remark. If $\sum u_i = 0$, the term, l_∞, drops out of (4.22) and one gets the following simpler expression for ω:

$$i\partial\bar{\partial}\phi^*\left(\sum_{k=1}^{d} \lambda_k \mathrm{Log}\, l_k\right). \tag{4.23}$$

The condition, $\sum u_i = 0$, has the following nice geometric interpretation: Let N be the subgroup of the d-dimensional torus

$$S^1 \times \cdots \times S^1 \qquad (d \text{ copies}) \tag{4.24}$$

defined by (2.13). If $\sum u_i = 0$, then by (2.3), N contains the diagonal subgroup of (4.24). Hence, by the principle of "reduction in stages", X is the reduced space associated with the Hamiltonian action of N/S^1 on $\mathbb{C}P^{d-1}$.

5. Let $\rho\colon T^n \to U(N+1)$ be a linear representation of T^n on $\mathbb{C}^{N+1}$ with weights, $\alpha_0, \cdots, \alpha_N$, ρ can be extended to a linear representation

$$\rho_{\mathbb{C}}\colon T^n_{\mathbb{C}} \to GL(N+1, \mathbb{C}), \tag{5.1}$$

and from this representation one gets an induced action

$$\tau_{\mathbb{C}}\colon T^n_{\mathbb{C}} \to PL(N+1, \mathbb{C}) \tag{5.2}$$

of $T^n_{\mathbb{C}}$ on $\mathbb{C}P^N$. Let W be a $T^n_{\mathbb{C}}$ orbit in $\mathbb{C}P^N$. Our first goal in this section will be to compute the Kaehler metric induced on W by the Fubini-Study metric on $\mathbb{C}P^N$. Let p be a point of W. For simplicity we will assume that $T^n_{\mathbb{C}}$ acts freely at p, and we will identify W with $T^n_{\mathbb{C}}$ by the evaluation map: $g \in T^n_{\mathbb{C}} \to \tau_{\mathbb{C}}(g)p$. One can also identify $T^n_{\mathbb{C}}$ with the linear space, $\mathbb{C}^2/2\pi i\mathbb{Z}^n$, by the exponential mapping, and by composing these two mappings, one gets a complex embedding: $\iota\colon \mathbb{C}^n/2\pi i\mathbb{Z}^n \to \mathbb{C}P^N$ whose image is W. Denoting by ω_{FS} the Fubini-Study form on $\mathbb{C}P^N$ we will prove

$$\iota^*\omega_{FS} = i\partial\bar{\partial}\mathrm{Log}\left(\sum_{i=0}^{N} c_i e^{2\alpha_i(x)}\right), \tag{5.3}$$

the c_i's being nonnegative constants.

Proof. Without loss of generality we can assume that $\rho_{\mathbb{C}}$ is a homomorphism of $T^n_{\mathbb{C}}$ into the diagonal subgroup of $GL(N+1, \mathbb{C})$. In other words for

$$z = x + iy \in \mathbb{C}^n/2\pi i\mathbb{Z}^n,$$

and $a = (a_0, \cdots, a_N) \in \mathbb{C}^N$,

$$\rho_{\mathbb{C}}(\exp z)a = (e^{\langle\alpha_0, x+iy\rangle}a_0, \cdots, e^{\langle\alpha_N, x+iy\rangle}a_N). \tag{5.4}$$

However, by [3, p. 45],

$$\omega_{FS} = i\partial\bar{\partial}\mathrm{Log}|z|^2. \tag{5.5}$$

Hence, letting $[a_0, \cdots, a_N]$ be the homogeneous coordinates of the point, p, from (5.4) one gets

$$\imath^*\omega_{FS} = i\partial\bar{\partial}\mathrm{Log}\left(\sum_{i=0}^{N} |a_i|^2 e^{2\langle\alpha_i, x\rangle}\right),$$

which is equal to (5.3) with $c_i = |a_i|^2$.

Now let X be the n-dimensional toric variety associated with the polytope (2.1), and let $\imath\colon X \to \mathbb{C}P^N$ be a $T^n_{\mathbb{C}}$ equivariant embedding. If M is the open $T^n_{\mathbb{C}}$ orbit in X, then its image with respect to $\imath$ is a $T^n_{\mathbb{C}}$ orbit in $\mathbb{C}P^N$, so the restriction of $\imath^*\omega_{FS}$ to M is still given by (5.3). Let us compare $\imath^*\omega_{FS}$ with the Kaehler form, ω, defined by (1.3). If we make the change of coordinates $\partial F/\partial x = y$ and $\partial G/\partial y = x$, F and G being the functions (4.22) and (4.9), then from (4.9) we get

$$\frac{\partial G}{\partial y_i} = \frac{1}{2}\left(\sum_{i=1}^{d} u_i \mathrm{Log}\, l_i + u\right)$$

with $u = \sum u_i$, hence

$$2\langle\alpha, x\rangle = \left\langle\alpha, \frac{\partial G}{\partial y}\right\rangle = \sum\langle\alpha, u_i\rangle\mathrm{Log}\, l_i + \langle\alpha, u\rangle.$$

By letting $d_\alpha = e^{\langle\alpha, u\rangle}$ this gives

$$e^{2\langle\alpha, x\rangle} = \phi^*\left(d_\alpha \prod l_i^{\langle\alpha, u_i\rangle}\right).$$

However, in consequence of (4.22),

$$e^{2F} = \phi^*\left(e^{l_\infty}\prod_{k=1}^{d} l_k^{\lambda_k}\right);$$

so we can rewrite the former expression in the form

$$e^{2\langle\alpha, k\rangle} = e^{2F}\phi^*\left(e^{-l_\infty}\prod_{k=1}^{d} l_k^{n_{k\alpha}}\right),$$

where

$$n_{k\alpha} = \langle\alpha, u_k\rangle - \lambda_k = l_k(\alpha). \tag{5.6}$$

Substituting this expression into (5.3), letting $k_\alpha = c_\alpha d_\alpha$, and summing over α, one gets:

$$\sum c_a e^{2\langle\alpha, x\rangle} = e^{2F}\phi^*(e^{-l_\infty}P)$$

with

$$P = \sum_{\alpha} k_\alpha \prod_{i=1}^{d} l_i^{n_{i\alpha}}. \tag{5.7}$$

Thus, from (4.2) and (5.3), it finally follows that

$$l^* \omega_{FS} = \omega + i\partial\overline{\partial}\phi^*(-l_\infty + \operatorname{Log} P). \tag{5.8}$$

Let $[\omega]$ and $\imath^\#[\omega_{FS}]$ be the DeRham cohomology classes in $H^2(X, \mathbb{R})$ associated with ω and $\imath^*\omega_{FS}$. As an application of (5.8) we will prove that

$$[\omega] = \imath^\#[\omega_{FS}], \tag{5.9}$$

if and only if the following three conditions hold:

(5.10)
(i) The vertices of Δ are in $\mathbb{Z}^n$.
(ii) If $k_\alpha \neq 0$, $\alpha \in \Delta$.
(iii) If α is a vertex of Δ, $k_\alpha \neq 0$.

Proof. We will first prove a variant of Theorem 4.3.

Theorem 5.1. *If* (5.9) *holds, there exists a smooth function, Q, on $\mathbb{R}^n$ such that*

$$\omega_{FS} = \omega + i\partial\overline{\partial}\phi^* Q. \tag{5.11}$$

Moreover, Q is unique up to an additive constant.

Proof. X is simply connected (see, for instance, [4]); so $H^1(X, \mathbb{R}) = 0$. Since X is Kaehler,

$$H^{1,0}(X) = H^{0,1}(X) = 0. \tag{5.12}$$

If (5.9) holds, there is a T^n-invariant one-form, μ, on X satisfying $\omega_{FS} = \omega + d\mu$. Let $\mu = \alpha + \overline{\alpha}$ with $\alpha \in \Omega^{0,1}(X)$. Since $d\mu$ is of bidegree (1,1), $\overline{\partial}\alpha = 0$; therefore, by (5.12) there is a T^n-invariant function, G, such that $\alpha = \overline{\partial}G$. Thus, $\imath^*\omega_{FS} - \omega = i\partial\overline{\partial}(2 \operatorname{Im} G)$. Since $\operatorname{Im} G$ is T^n-invariant, by [12] there exists a smooth function, Q, on $\mathbb{R}^n$ such that $2 \operatorname{Im} G = \phi^* Q$.

To prove the uniqueness of Q consider the restriction of $\imath^*\omega_{FS} - \omega$ to the open $T^n_{\mathbb{C}}$ orbit in X. On this orbit

$$\imath^*\omega_{FS} - \omega = \frac{i}{4}\sum \frac{\partial^2 \phi^* Q}{\partial x_i \partial x_j} dz_i \wedge d\overline{z}_i;$$

so $\phi^* Q$ is well-determined up to a linear function of x. However, since $\phi^* Q$ is bounded, it is actually well determined up to a constant. q.e.d.

Comparing (5.11) with (5.8), we conclude that for (5.9) to hold, $\operatorname{Log} P$ must be a smooth function on Δ. However, it is clear from (5.7) that P is a smooth fucntion on Δ if and only if it is a polynomial, i.e., if and only if $n_{i\alpha}$ is a nonnegative integer for all i, α. By (5.6), $n_{i\alpha}$ is an integer if and only if λ_i is an integer, and $n_{i\alpha}$ is nonnegative if and only if $l_i(\alpha) \geq 0$. But $l_i(\alpha) \geq 0$ for all i if and only if $\alpha \in \Delta$.

Finally, $\operatorname{Log} P$ is a smooth function on Δ if and only if P is positive on Δ. Suppose P is zero at some point, $q \in \Delta$. Then each summand of (5.7) has to be zero at q, and hence on the face, F, of Δ containing q. Let us consider the worst case scenario: $q = \beta = a$ vertex of F. Suppose the term

$$k_\alpha \prod_{k=1}^{d} l_k(\beta)^{n_{k\beta}} \tag{5.13}$$

is *not* zero. Let the $(n-1)$-dimensional faces of Δ meeting at β be those defined by the equations, $l_i = 0$, $i = i_1, \cdots, i_n$. If (5.13) is *not* zero, then $n_{i\alpha} = 0$ for $i = i_1, m\cdots, i_n$, and hence $l_i(\alpha) = 0$ for $i = i_1, \cdots, i_n$ by (5.6); i.e., $\beta = \alpha$. Therefore, by (5.13) a necessary and sufficient condition for P to be positive on Δ is that k_α be positive for every vertex, $\alpha \in \Delta$.

6. Let (M, ω) be a compact connected Kaehler manifold and let

$$\tau : S^1 \to \operatorname{Diff}(M) \tag{6.1}$$

be an effective action of S^1 on M by complex-analytic diffeomorphisms, which preserves ω and is Hamiltonian; i.e., has a global moment map

$$\phi : M \to \mathbb{R}. \tag{6.2}$$

Let Y be a connected component of the fixed point set of M. It is clear that Y is a complex submanifold of M (since τ acts complex analytically), and hence is also a symplectic submanifold of M. Moreover, the restriction of ϕ to Y is equal to a constant, c. Suppose in addition that Y is of complex codimension one. In this case we claim *that c has to be an extremal value of ϕ, i.e., either a maximum or a minimum, and that $Y = \phi^{-1}(c)$.*

Proof. We will first prove

Theorem 6.1. *Let $p \in Y$ and let $\mathbb{L}_p$ be the fiber of the normal bundle at p. Then the weight of the isotropy representation of τ on $\mathbb{L}_p$ is ± 1.*

Proof. If the weight were m, $m \neq \pm 1$, the mth roots of unity would be in the kernel of the homomorphism, (6.1), contradicting the assumption that τ is effective. q.e.d.

For simplicity we assume that the weight of the isotropy representation is $+1$. Then, by the equivariant Darboux theorem (see, for instance, [8, §22]) there exist Darboux coordinates $x_1, y_1, \cdots, x_n, y_n$ centered at p so that in these coordinates $\phi = \frac{1}{2}(x_1^2 + y_1^2) + c$. Thus, in particular, c is a local minimum of ϕ. However, by the Atiyah connectivity theorem, $\phi^{-1}(c)$ is connected. Hence $\phi^{-1}(c) = Y$, and Y is the set where ϕ achieves its global minimum. q.e.d.

Next we will prove that for any $p \in Y$ there exist an S^1-invaraint open neighborhood, U, of p, and complex coordinates, $z_1, \cdots, z_n$, on U such that $U \cap Y$ is the set where $z_1 = 0$, and such that

$$\tau_\theta^* z_1 = e^{i\theta} z_1 \tag{6.3}$$

and

$$\tau_\theta^* z_i = z_i, \quad \text{for } i > 1. \tag{6.4}$$

Proof. It is clear that there exists a complex coordinate system with all these properties except, perhaps, (6.3). However, by Lemma 6.1,

$$\tau_\theta^*(dz_1)_q = e^{i\theta}(dz_1)_q$$

for $q \in U \cap Y$; so if one replaces z_1 by $(1/2\pi)\int_0^{2\pi} e^{-i\theta}\tau_\theta^* z_1 \, d\theta$, the above coordinate system will have the property (6.3) as well. q.e.d.

Replacing ϕ by $\phi - c$ we can assume that $\phi = 0$ on Y and $\phi > 0$ on $M - Y$. We will now prove the following elementary but useful fact.

Theorem 6.2. *The cohomology class of the* $(1, 1)$ *form*

$$\mu = \frac{1}{2\pi i}\partial\bar{\partial}\operatorname{Log}\phi \tag{6.5}$$

is the dual cohomology class to Y *in* $H^2(M, \mathbb{R})$.

Proof. Let $p \in Y$ and let $(U, z_1, \cdots, z_n)$ be a coordinate system centered at p having the proeprties (6.3)–(6.4). The Hessian of ϕ is nonzero at all points of $U \cap Y$, and ϕ is S^1-invariant; so, in terms of the above coordinates,

$$\phi = |z_1|^2 h, \tag{6.6}$$

h being a smooth function in the variables, $|z_1|^2, z_2, \bar{z}_2, \cdots, z_n, \bar{z}_n$, which is positive everywhere on U. Thus the restriction of (6.5) to U can be written in the form

$$\mu = \frac{i}{2\pi}\bar{\partial}\partial(\operatorname{Log}|z_1|^2 + \operatorname{Log} h) = \frac{1}{2\pi}d(\partial \operatorname{Log} h). \tag{6.7}$$

Now let $\{U_i, i = 1, \cdots, N\}$ be a good cover of Y by open S^1-invariant subsets of M, and let $(z_{1,i}, z_{2,i}, \cdots, z_{n,i})$ be a system of coordinates on U_i satisfying (6.3)–(6.4).

Letting

$$\phi = |z_{1i}|^2 h_i, \tag{6.8}$$

from (6.7) we get

$$\mu = d\alpha, \quad \text{on } U_i, \tag{6.9}$$

where

$$\alpha_i = (i/2\pi)\partial \operatorname{Log} h_i.$$

Thus, on $U_i \cap U_j$,

$$\alpha_i - \alpha_j = \frac{1}{2\pi i}\partial \operatorname{Log}|f_{ij}|^2, \tag{6.10}$$

by the identity (6.8), where

$$f_{ij} = z_{1i}/z_{1j}. \tag{6.11}$$

Since $U_i \cap U_j$ is contractible, there exists a single-valued determination of $\operatorname{Log} f_{ij}$; so we can rewrite (6.10) in the form

$$\alpha_i - \alpha_j = \frac{1}{2\pi i} d \operatorname{Log} f_{ij}. \tag{6.12}$$

We recall now that there is a holomorphic line bundle

$$\mathbb{L} \to M \tag{6.13}$$

canonically associated with the hypersurface, Y. This bundle can be defined either by taking its transition functions to be the functions (6.11) or by requiring that it have the following two properties:

(i) The restriction of $\mathbb{L}$ to Y is the normal bundle of Y.
(ii) The restriction of $\mathbb{L}$ to $M - Y$ is trivial.

Moreover, it can be characterized topologially by the property that its Chern class is the dual class in $H^2(M, \mathbb{Z})$ to the homology class [Y], in $h_{2n-2}(M, \mathbb{Z})$. (For details, see [6, pp. 129–148].) However, by (6.9) and (6.12) the Chern class of $\mathbb{L}$ in $H^2(M, \mathbb{R})$ is the cohomology class of the form (6.5). q.e.d.

Now let X be the toric variety associated with the polytope, Δ, and let u_i be, as in (2.1), the normal vector to the ith face of Δ. Let

$$\{tu_i, \ t \in \mathbb{R}/2\pi\mathbb{Z}\} \tag{6.14}$$

be the one-parameter subgroup of T^n generated by u_i. The moment map associated with the action of this group is $l_i \circ \phi$ which, by (1.2), takes its minimum on the set

$$\phi \circ l_i = 0. \tag{6.15}$$

We will denote this set by X_i. It is the pre-image in X of the ith $(n-1)$-dimensional face of Δ, and is a complex submanifold of X of codimension 1. Let c_i be the cohomology class in $H^2(X, \mathbb{R})$ dual to the homology class $[X_i]$ in $H_{2n-2}(X, \mathbb{R})$. By Theorem 6.2,

$$c_i = [\mu_i], \tag{6.16}$$

where

$$\mu_i = (i/2\pi)\partial\bar{\partial}\operatorname{Log}\phi^* l_i; \tag{6.17}$$

and by applying this result to (1.3) one gets:

Theorem 6.3. *Let ω be the Kaehler form on X, and $[\omega]$ its cohomology class. Then*

$$\frac{[\omega]}{2\pi} = -\sum_{i=1}^{d} \lambda_i c_i. \tag{6.18}$$

Remark. For an alternative proof of (6.18), based on the Duistermaat-Heckman theorem, see [7, Lecture 2].

We will now discuss some applications of (6.18). In these applications we will make use of the following:

Theorem 6.4. *The symplectic volume of X, $\int_X \exp(\omega)$, is equal to $(2\pi)^n$ times the Euclidean volume of Δ.*

Proof. Let M be the open $T^n_{\mathbb{C}}$ orbit in X, and, by fixing a base point on M, let us identify M with $\mathbb{C}^n/2\pi i\mathbb{Z}^n$. By (4.2) the restriction of ω to M is $\sum(\partial^2 F/\partial x_i \partial x_j)\, dx_i \wedge dy_j$; so

$$\frac{\omega^n}{n!} = \det\left(\frac{\partial^2 F}{\partial x_i \partial x_j}\right) dx \wedge dy.$$

Thus, by Fubini's theorem, the integral of $\exp\omega$ over X is the product of the integral of dy over $\mathbb{R}^n/2\pi\mathbb{Z}^n$ (which is just $(2\pi)^n$) and the integral

$$\int_{\mathbb{R}^n} \det\left(\frac{\partial^2 F}{\partial x_i \partial x_j}\right) dx. \tag{6.19}$$

However, by (4.4), the map

$$x + iy \to \partial F/\partial x \tag{6.20}$$

is the moment map (2.24), and its restriction to $\mathbb{R}^n$ maps $\mathbb{R}^n$ diffeomorphically onto the interior of Δ. Moreover, since F is strictly plurisubharmonic, $\det(\partial^2 F/\partial x_i \partial x_j) > 0$. Thus by the formula for the change of coordinates in elementary calculus, (6.19) is just the volume of Δ. q.e.d.

Recall now that Δ is defined by the system of inequalities:

$$\langle x, u_i \rangle \geq \lambda_i, \qquad i = 1, \cdots, d,$$

u_i being the "inward-pointing" normal to the ith $(n-1)$-dimensional face. Let $v_i = -u_i$, the "outward-pointing" normal to this face, and let $s_i = -\lambda_i$. Then Δ can also be defined by the inequalities:

$$\langle x, v_i \rangle \leq s_i, \qquad i = 1, \cdots, d. \tag{6.21}$$

Let $v(s_1, \cdots, s_d)$ be the Euclidean volume of this set. From the two previous theorems, one gets the following corollary:

Theorem 6.5. *The pairing of the cohomology class,* $\exp(\sum s_i c_i)$, *with the orientation cycle,* $[X]$, *in* $H^{2n}(X, \mathbb{R})$, *is equal to the volume,* $v(s_1, \cdots, s_d)$, *of the set* (6.21); *i.e.,*

$$v(s_1, \cdots, s_d) = \exp\left(\sum s_i c_i\right)[X]. \tag{6.22}$$

We recall (see [4, Proposition 11.4) that the Chern classes of X are just the elementary symmetric polynomials in $c_1, \cdots, c_d$. In other words the rth Chern class is the coefficient of t^r in the expression

$$\prod_{i=1}^{d}(1 + tc_i). \tag{6.23}$$

As a consequence of this [4, 11.7] the Todd class of X can be expressed in terms of the c_i's as

$$\tau = \prod_{i=1}^{d} \frac{c_i}{1 - e^{-c_i}}. \tag{6.24}$$

Now suppose that $\omega/(2\pi)$ is an integer cohomology class, or, equivalently, that $s_1, \cdots, s_d$ are integers. Let $\mathbb{L}_i \to X$ be the holomorphic line bundle associated with the hypersurface, X_i, and let $\mathbb{L} = \mathbb{L}_1^{s_1} \otimes \cdots \otimes \mathbb{L}_d^{s_d}$. By Lemma 6.1 the Chern class of $\mathbb{L}$ is:

$$c(\mathbb{L}) = \sum_{i=1}^{d} s_i c_i = \frac{1}{2\pi}[\omega]. \tag{6.25}$$

Let $\mathscr{L}$ be the sheaf of holomorphic sections of $\mathbb{L}$. Then by the Hirzebruch-Riemann-Roch formula,

$$\sum_{i=0}^{n}(-1)^i \dim H^i(X, \mathscr{L}) = \tau c(\mathbb{L}))[X]. \tag{6.26}$$

Substituting (6.25) in (6.26), one gets for the sum on the left-hand side of (6.26): $(\exp(\sum s_i c_i)\tau(c_1, \cdots, c_d))[X]$. Let us rewrite this as follows. Let $\tau(\partial/\partial s)$ be the constant coefficient differential operator (of infinite order)

$$\prod_{i=1}^{d} \frac{\partial}{\partial s_i}\left(1 - \exp\left(-\frac{\partial}{\partial s_i}\right)\right)^{-1}.$$

Applying this operator to $v(s_1, \cdots, s_d)$ (which is legitimate since v is a polynomial) we obtain from (6.22) $(\exp(\sum s_i c_i)\tau(c_1, \cdots, c_d))[X]$ which is exactly the right-hand side of (6.27). This shows that the right-hand side of (6.27) has a simple "combinatorial" interpretation. It turns out [4, 11.12] that the same is true of the left-hand side. Namely $H^i(X, \mathscr{L}) = 0$ if $i > 0$ by the Kodaira vanishing theorem, and

$$\dim H^\circ(X, \mathscr{L}) = \#(\Delta \cap \mathbb{Z}^n)$$

by a fairly easy computation. Thus (6.26) reduces to

$$\#(\Delta \cap \mathbb{Z}^n) = \tau(\partial/\partial s)v(s). \tag{6.27}$$

This is the Khovanskii-Pukhlikov identity which we referred to in the introduction.

Remark. Since (6.27) is a combinatorial statement, one would expect to be able to prove it by combinatorial means; and this, in fact, is what Khovanskii and Pukhlikov did in [11]. We suspect, however, that this may have been an ex post facto development, and that they may have been led to (6.27) by arguments similar to those above. (For some recent refinements of (6.27) see [10], [14], and [15].)

Next we will turn to the second application of (6.18) which we mentioned in the introduction. Let Δ be a polygon in the plane having the properties described in the first paragraph of §2. (In particular let all the vertices of Δ be nonsingular.) Given an edge, F_i, of Δ let p_i and p_{i+1} be its end points. Since p_i is nonsingular, there exists a basis, $\{w_{i,1}, w_{i,2}\}$, of $\mathbb{Z}^2$ such that the edges intersecting in p_i are on the rays $p_i + tw_{i,1}$ and $p_i + tw_{i,2}$, $t \geq 0$. Therefore, since p_i and p_{i+1} share a common edge, $w_{i+1,1} = -w_{i,2}$, and hence the matrix which relates $\{w_{i,1}, w_{i,2}\}$ to $\{w_{i+1,1}, w_{i+1,2}\}$ has the form

$$\begin{pmatrix} 0 & k_i \\ -1 & m_i \end{pmatrix},$$

k_i and m_i being integers. However, the determinant of this matrix is 1; so $k_i = 1$ and this matrix reduces to:

$$M_i = \begin{pmatrix} 0 & 1 \\ -1 & m_i \end{pmatrix}. \tag{6.28}$$

The matrices (6.28) cannot be completely arbitrary. Indeed, since the end point of F_d coincides with the initial point of F_1, $M_1 \cdots M_d = I$; and, in addition, the M_i's have to satisfy a "topological" constraint corresponding to the fact that the frame, $\{w_{i1}, w_{i2}\}$, rotates once about the origin as i goes from 1 to d. Conversely, the problem of constructing polygons with the properties described in §2 can be more or less reduced to the problem of finding sequences of integers, m_1, $i = 1, \cdots, d$, for which the matrices, M_i, have the properties above. (For details see [2, Chapter VI, §5.1].)

Now let X be the two-dimensional complex toric manifold associated with Δ, and let X_i, $i = 1, \cdots, d$, be the complex one-dimensional submanifolds corresponding to the F_i's. Audin shows (loc. cit.) that the numbers, m_i, have the following geometric interpretation:

Theorem 6.6. *$-m_i$ is the self-intersection number of X_i in X.*

This result completely determines the cohomology ring of X since it is clear that the intersection number $\#(X_i \cap X_j)$ is one, if $i \neq j$, and if F_i and F_j are adjacent and zero otherwise.

I will now describe a generalization of this result to n dimensions. Let X be the n-dimensional toric manifold associated with the polytope (2.1), and let X_i, $i = 1, \cdots, d$, be the complex hypersurfaces corresponding to the $(n-1)$-dimensional faces of this polytope. Let $n_1, \cdots, n_d$ be nonnegative integers satisfying $n = \sum_{i=1}^d n_i$, and consider the intersection number

$$\#(X_1^{n_1} \cap \cdots \cap X_d^{n_d}), \tag{6.29}$$

where $X_i^{n_i}$ is the n_i-fold intersection of X_i with itself.

Theorem 6.7. *The intersection number* (6.29) *is equal to*

$$n_1! \cdots n_d! a_{n_1 \cdots n_d}, \tag{6.30}$$

where $a_{n_1 \cdots n_d}$ is the coefficient of $s_1^{n_1} \cdots s_d^{n_d}$ in the polynomial, $v(s_1, \cdots, s_d)$.

Proof. By (6.22) we have

$$\left(\frac{\partial}{\partial s_1}\right)^{n_1} \cdots \left(\frac{\partial}{\partial s_d}\right)^{n_d} v(s_1, \cdots, s_d) = c_1^{n_1} \cdots c_d^{n_d}[X].$$

Remark. We leave it as an amusing exercise to deduce Audin's result from Theorem 6.7!

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